Fuzzy Actions and their Continuum Limits

A.P. Balachandran*, X. Martin† and Denjoe O'Connor†

*Physics Department, Syracuse University,
Syracuse, N.Y., 13244-1130, U.S.A.
† Depto. de Física, Cinvestav,
Apartado Postal 70-543, México D.F. 07300, México.

Previously matrix model actions for "fuzzy" fields have been proposed using non-commutative geometry. They retained "topological" properties extremely well, being capable of describing instantons, θ–states, the chiral anomaly, and even chiral fermions with no "doubling". Here, we demonstrate that the standard scalar and spinor actions on a d–dimensional manifold are recovered from such actions in the limit of large matrices if their normalizations are correctly scaled as the limit is taken.

I. INTRODUCTION

In contrast to conventional lattice discretizations, fuzzy physics [1–6] regulates quantum fields on a manifold $\mathcal{M}$ by quantizing the latter, treating it as a phase space. This method works because quantization implies a short distance cut–off: the number of states in a volume V on $\mathcal{M}$ is infinite in classical physics, but becomes about $V/\tilde{\hbar}^d$ on quantization, if $\tilde{\hbar}$ is the substitute for Planck's constant and $\mathcal{M}$ has dimension d. The classical limit $\tilde{\hbar} \to 0$ is then the continuum limit of interest.

In past research, [7–16] several authors have explored this novel approach to discrete physics and established that it can correctly reproduce continuum topological features like instantons, θ–states and the axial anomaly. Even chiral fermions can be formulated without duplication [14]. When $\mathcal{M}$ is the two sphere S^2, this formulation shares common features with the Ginsparg-Wilson approach [15].

Proposals have also been made in previous work for certain novel fuzzy actions. They fulfill instanton-like bounds when such exist in the continuum and in all cases are compatible with the known scaling properties of the latter. They were conjectured to have correct continuum limits as well. This paper verifies those conjectures for the scalar and spinor fields. Gauge theories will be examined elsewhere, but the correctness of the conjectures in all instances looks plausible. The present work in this manner goes toward establishing that the actions of [13,14] in addition to retaining important topological features, also have the correct continuum limits. There is thus good reason to explore them further as alternatives to conventional lattice models.

In section II, we review the motivations and structure of the proposed actions and indicate which of them we will study in this paper. Introductory developments and useful asymptotic estimates involving heat kernels methods are covered in sections III, IV and the Appendix. The remaining two sections successfully employ this material to establish the continuum limits. The concluding remarks are in section VII.

II. A REVIEW

Fuzzy physics is based on non-commutative geometry. In the continuum limit $\tilde{\hbar} \to 0$, it is thus appropriate to compare its action with those of conventional continuum physics written in the

language of non-commutative geometry. The proper fuzzy actions were in fact inferred by their resemblance to the latter.

A central role is played in non-commutative geometry by the Dirac operator. Let $\mathcal{D}$ be this operator for a manifold $\mathcal{M}$ of dimension d. For a scalar field ϕ on $\mathcal{M}$, the free action according to Connes is

$$S(\phi) = \mathrm{Tr}^{+}\left(\frac{1}{|\mathcal{D}|^{d}}[\mathcal{D},\phi]^{\dagger}[\mathcal{D},\phi]\right), \tag{2.1}$$

where ϕ and $\mathcal{D}$ are regarded as operators on the Hilbert space of spinors and Tr^{+} is the Dixmier trace [17]. It can be explained as follows. Let A be the operator within Tr^{+} in (2.1) and E^2 be the eigenvalues of $\mathcal{D}^{\dagger}\mathcal{D} \equiv |\mathcal{D}|^2$. If $\mathrm{Tr}_{|E|}(A)$ is the trace of A up to eigenvalues $|E|$ of $|\mathcal{D}|$, then

$$\mathrm{Tr}^{+}(A) = \lim_{|E|\to\infty}\left(\frac{1}{\ln|E|}\mathrm{Tr}_{|E|}(A)\right). \tag{2.2}$$

Connes shows that

$$S(\phi) = \int_{\mathcal{M}} |\nabla\phi|^2 d^d\mathbf{n}, \tag{2.3}$$

where ∇ is the covariant derivative operator in $\mathcal{D}$ and $d^d\mathbf{n}$ is the volume form on $\mathcal{M}$. In case $\mathcal{D}$ has zero modes, we will simply exclude the corresponding eigenspace from all traces.

There is a similar formulation of actions $S(\psi)$ for free spinors:

$$S(\psi) = \mathrm{Tr}^{+}\left(\frac{1}{|\mathcal{D}|^{d}}\psi^{\dagger}\mathcal{D}\psi\right). \tag{2.4}$$

Actions appropriate for gauge theories will not be examined in this paper.

Under the scaling $\mathcal{D} \to \lambda\mathcal{D}$, the response of these actions is

$$S(\phi) \to \lambda^{2-d}S(\phi),\ \ S(\psi) \to \lambda^{1-d}S(\psi). \tag{2.5}$$

The critical dimensions for ϕ and ψ to have scale invariant actions are thus $d = 2$ and $d = 1$ respectively. The latter will not be of interest to us since a line or a circle is not symplectic and therefore cannot be quantized.

There is an alternative form of great interest for these actions. Let

$$\varepsilon = \mathrm{sign}(\mathcal{D}) \equiv \frac{\mathcal{D}}{|\mathcal{D}|} \tag{2.6}$$

be the sign of the Dirac operator. Then, as we will establish in subsequent sections,

$$S(\phi) = \frac{d-1}{d}\mathrm{Tr}^{+}\left(\frac{1}{|\mathcal{D}|^{d-2}}[\varepsilon,\phi]^{\dagger}[\varepsilon,\phi]\right), \tag{2.7}$$

$$S(\psi) = \frac{d-1}{d}\mathrm{Tr}^{+}\left(\frac{1}{|\mathcal{D}|^{d-1}}\psi^{\dagger}\varepsilon\psi\right). \tag{2.8}$$

In fuzzy physics too, there is a Dirac operator with a proper continuum limit. It is therefore natural to model its action on (2.7,2.8) or on (2.1,2.4). But as the fuzzy action involves only finite dimensional Hilbert spaces, Tr^{+} must be substituted by an approximately normalised version of the normal trace. In that case, the fuzzy versions of (2.7,2.8) and (2.1,2.4) are no longer the same. There are powerful topological reasons for preferring (2.7,2.8) over (2.1,2.4) as models for fuzzy versions. As has been shown elsewhere, only these versions naturally fulfill instanton–like bounds and lead to representations of θ–states. Our task is thus to establish that fuzzy models based on (2.7,2.8) have the correct limits.

By way of orientation, we now give a brief account of the fuzzy sphere and its Dirac operator. Our discussion thereafter is a great deal more general, but it is a useful illustration to keep in mind.

A. The fuzzy sphere

The sphere S^2 is a submanifold of $\Re^3$ given by

$$S^2 = \{\mathbf{n} \in \Re^3 : \sum_{i=1}^{3} n_i^2 = 1\}. \tag{2.9}$$

If $\hat{n}_i$ are the coordinate functions on S^2, $\hat{n}_i(\mathbf{n}) = n_i$, then $\hat{n}_i$ commute and the algebra $\mathcal{A}$ of smooth functions they generate is commutative.

In contrast, the operators x_i describing S_F^2 are non-commutative, their commutators being given by

$$[x_i, x_j] = \frac{i\epsilon_{ijk}x_k}{\sqrt{l(l+1)}}, \ \sum_{i=1}^{3} x_i^2 = \mathbf{1}, \ l \in \{\frac{1}{2}, 1, \frac{3}{2}, \ldots\}. \tag{2.10}$$

The x_i approach $\hat{n}_i$ as $l \to \infty$. If $L_i = x_i\sqrt{l(l+1)}$, then, from (2.10), it is clear that the L_i give the irreducible representation (IRR) of $SU(2)$ with angular momentum l. L_i or x_i generate the algebra $A_l = M_{2l+1}$ of $(2l+1) \times (2l+1)$ matrices.

A scalar product on A_l is $<\xi, \eta> = \mathrm{Tr}(\xi^\dagger \eta)$. A_l acts on this Hilbert space by left– and right– multiplications, giving rise to the left– and right– regular representations $A_l^{L,R}$ of A_l. For each $a \in A_l$, we thus have two operators $a^{L,R} \in A^{L,R}$ acting on $\xi \in A_l$ according to $a^L\xi = a\xi$ and $a^R\xi = \xi a$. Note that $a^L b^L = (ab)^L$ but that $a^R b^R = (ba)^R$. We assume by convention that elements of A_l^L are to be identified with fields or functions on S^2. This identification is coherent with the fact that fields or functions on S^2 come from $\mathcal{A}$.

Of particular interest are the angular momentum operators. There are two kinds of angular momenta $L_i^{L,R}$ for S_F^2, while the orbital angular momentum operator, which should annihilate $\mathbf{1}$ is $\mathcal{L}_i = L_i^L - L_i^R$. $\vec{\mathcal{L}}$ plays the role of the continuum $-i(\mathbf{r} \times \nabla)$. The "position" operators are not proportional to $\mathcal{L}_i$, but are instead $L_i^L/\sqrt{l(l+1)}$.

The Dirac operator on S_F^2 is

$$D = \sigma \cdot \mathcal{L} + \mathbf{1}. \tag{2.11}$$

It acts on $A_l \otimes \mathcal{C}^2 = \{(\psi_1, \psi_2) : \psi_i \in A_l\}$.

D admits a chirality γ with which it anticommutes only after state vectors of maximum total angular momentum $j = 2l + 1/2$ have been projected out. This projection has no effect on our analysis however and will therefore be ignored in the following.

B. Preliminaries towards the Continuum

Let M_l be the fuzzy space for a manifold $\mathcal{M}$ of dimension d and let D be its Dirac operator. The sign of D is

$$\epsilon = \frac{D}{|D|}, \tag{2.12}$$

where it is understood that 0–modes of D are projected out of the Hilbert space. The scalar and spinor field actions modeled on (2.7) and (2.8) are then

$$S_l(\phi) = \frac{1}{g_l}\mathrm{Tr}_{\mathcal{H}^l}\left(\frac{1}{|D|^{d-2}}[\epsilon, \phi]^\dagger[\epsilon, \phi]\right), \tag{2.13}$$

$$S_l(\psi) = \frac{1}{g_l}\mathrm{Tr}_{\mathcal{H}^l}\left(\frac{1}{|D|^{d-1}}\psi^\dagger \epsilon \psi\right), \tag{2.14}$$

where g_l is a logarithmically divergent (for $l \to \infty$) normalization, choosen such that

$$\lim_{l\to\infty} \frac{1}{g_l} \mathrm{Tr}_{\mathcal{H}^l} \left(\frac{1}{|D|^d} \right) = \text{Volume of}(\mathcal{M}) \tag{2.15}$$

Note that the scalar action $S(\phi)$ is scale invariant in the critical dimension $d = 2$, which includes the case of the fuzzy sphere described above.

Operators and traces are associated with a finite dimensional Hilbert space $\mathcal{H}^l$. For S_F^2, the dimension of $\mathcal{H}^l$ is $2(2l+1)^2$. With increasing l, we have a family of nested Hilbert spaces $\ldots \subset \mathcal{H}^l \subset \mathcal{H}^{l+1} \subset \ldots$ with a limiting Hilbert space $\mathcal{H}^\infty$. Now, as $l \to \infty$, D goes over to the continuum Dirac operator when both are restricted to any finite–dimensional subspace. We can thus think of operators and traces in $S(\phi)$ and $S(\psi)$ as being associated with $\mathcal{H}^\infty$. As for ϕ and ψ, we require them to become smooth continuum fields as $l \to \infty$. The exact behavior we need is better stated in a coherent state basis which will be done below.

We will consider the following more general forms of actions which include (2.13) and (2.14) as special cases:

$$S(\phi_0, \phi_1, \ldots, \phi_n) = \frac{1}{g_l} \mathrm{Tr}_{\mathcal{H}^l} \left(\frac{1}{|D|^p} \phi_0 [\epsilon, \phi_1] \ldots [\epsilon, \phi_n] \right), \tag{2.16}$$

$$S(\psi_1, \psi_2) = \frac{1}{g_l} \mathrm{Tr}_{\mathcal{H}^l} \left(\frac{1}{|D|^p} \psi_1^\dagger \epsilon \psi_2 \right), \tag{2.17}$$

where ϕ_i and ψ_i are respectively fuzzy scalar and spinor fields.

Let $\Lambda(l)$ be a cut–off eigenvalue of $|\mathcal{D}|$ which goes to ∞ with l. Then $1/|\mathcal{D}_l|$ is defined as $1/|\mathcal{D}|$ whose modes with eigenvalue smaller than $1/|\Lambda(l)|$ are projected out or exponentially suppressed, and $\varepsilon_l = \mathcal{D}/|\mathcal{D}_l|$. We can then approximate the above actions by

$$S_l(\phi_0, \phi_1, \ldots, \phi_n) = \frac{1}{g_l} \mathrm{Tr}_{\mathcal{H}^\infty} \left(\frac{1}{|\mathcal{D}_l|^p} \phi_0 [\varepsilon_l, \phi_1] \ldots [\varepsilon_l, \phi_n] \right), \tag{2.18}$$

$$S_l(\psi_1, \psi_2) = \frac{1}{g_l} \mathrm{Tr}_{\mathcal{H}^\infty} \left(\frac{1}{|\mathcal{D}_l|^p} \psi_1^\dagger \varepsilon_l \psi_2 \right), \tag{2.19}$$

as argued in the Appendix. We focus on these actions in what follows as they are much easier to study.

The fuzzy spaces under consideration will be assumed to admit coherent state representations. That is the case for S_F^2, $\mathcal{C}P_F^N$ and many others. Let $|\mathbf{n}, l\rangle$ denote the coherent state basis for $\mathcal{H}^l$, where $\mathbf{n} \in \mathcal{M}$, with the normalization and conventions

$$\langle \mathbf{n}, l | \mathbf{n}', l \rangle = \delta^d(\mathbf{n}, \mathbf{n}') + o(1) \tag{2.20}$$

$$\int_{\mathcal{M}} d^d\mathbf{n} |\mathbf{n}, l\rangle\langle \mathbf{n}, l| = \mathbf{1}. \tag{2.21}$$

Here δ^d is the δ distribution on $\mathcal{M}$, $d^d\mathbf{n}$ is the volume form and $\mathbf{1}$ the identity operator. Then the assumptions made on the fields are

$$\langle \mathbf{n}, l | \phi_i | \mathbf{n}', l \rangle = \phi_i(\mathbf{n}) \delta^d(\mathbf{n}, \mathbf{n}') + o(1) \tag{2.22}$$

$$\langle \mathbf{n}, l | (\psi_i)_r | \mathbf{n}', l \rangle = (\psi_i)_r(\mathbf{n}) \delta^d(\mathbf{n}, \mathbf{n}') + o(1) \tag{2.23}$$

as $l \to \infty$. On the right hand side, ϕ_i and ψ_i denote fields on the continuum manifold $\mathcal{M}$, and $(\psi)_r$ is the r-th component of the spinor ψ. The use of the same symbols ϕ_i and ψ_i on both sides of (2.22) and (2.23) is not correct as they are matrices on the left hand side, but we retain this notation for convenience.

With (2.22), one has to leading order

$$\langle \mathbf{n}, l | [\epsilon, \phi_i] | \mathbf{n}', l \rangle = \mathcal{D}\mathcal{G}_l(\mathbf{n}, \mathbf{n}') \left(\phi_i(\mathbf{n}) - \phi_i(\mathbf{n}') \right) + o(1) \tag{2.24}$$

where $\mathcal{G}_l$ is the kernel of $1/|\mathcal{D}_l|$,

$$\mathcal{G}_l(\mathbf{n}, \mathbf{n}') = <\mathbf{n}| \frac{1}{|\mathcal{D}_l|} |\mathbf{n}'>, \tag{2.25}$$

$|\mathbf{n}>= |\mathbf{n}, \infty>$ being the state vectors localized at $\mathbf{n} \in \mathcal{M}$. Thus, it appears that the calculation of the limit $l \to \infty$ requires finding the behavior of $\mathcal{G}_l$ for large l.

III. PRELIMINARY ESTIMATES

The necessary asymptotic estimates are best approached by adapting heat kernel methods. The relevant heat kernel is

$$\mathcal{K}(\mathbf{n}, \mathbf{n}', t) = < \mathbf{n}| \mathrm{e}^{-\mathcal{D}^2 t} |\mathbf{n}' > . \tag{3.1}$$

It fulfills the heat equation

$$\partial_t \mathcal{K}(\mathbf{n}, \mathbf{n}', t) + \int d^d\mathbf{n}'' < \mathbf{n}|\mathcal{D}^2|\mathbf{n}''> \mathcal{K}(\mathbf{n}'', \mathbf{n}', t) = 0 \tag{3.2}$$

with the initial condition

$$\mathcal{K}(\mathbf{n}, \mathbf{n}', 0) = \delta^d(\mathbf{n}, \mathbf{n}'). \tag{3.3}$$

The operator $1/|\mathcal{D}_l|$, or more generally $1/|\mathcal{D}_l|^\theta$, can be expressed in terms of this kernel, by using

$$\frac{1}{|\mathcal{D}_l|^\theta} = \frac{1}{\Gamma(\frac{\theta}{2})} \int_0^\infty \frac{dt}{t} t^{\theta/2} \mathrm{e}^{-\mathcal{D}_l^2 t}. \tag{3.4}$$

Eigenvalues of $\mathcal{D}^2$ exceeding $\Lambda(l)^2$ are cut off or exponentially suppressed in $1/\mathcal{D}_l^2$. So we can approximate the right hand side in (3.4) by replacing $\mathcal{D}_l^2$ by $\mathcal{D}^2$ and restricting integration in t to $t \geq T(l) \approx 1/\Lambda(l)^2$. We thus have that

$$\frac{1}{|\mathcal{D}_l|^\theta} \simeq \frac{1}{\Gamma(\frac{\theta}{2})} \int_{T(l)}^\infty \frac{dt}{t} t^{\theta/2} \mathrm{e}^{-\mathcal{D}^2 t} \tag{3.5}$$

$$\mathcal{G}_l^\theta(\mathbf{n}, \mathbf{n}') := < \mathbf{n}| \frac{1}{|\mathcal{D}_l|^\theta} |\mathbf{n}' > \frac{1}{\Gamma(\frac{\theta}{2})} \int_{T(l)}^\infty \frac{dt}{t} t^{\theta/2} \mathcal{K}(\mathbf{n}, \mathbf{n}', t). \tag{3.6}$$

Furthermore, if T_0 is a (small) fixed number larger than $T(l)$, one can write

$$\mathcal{G}_l^\theta(\mathbf{n}, \mathbf{n}') = \frac{1}{\Gamma(\frac{\theta}{2})} \left(\int_{T(l)}^{T_0} + \int_{T_0}^\infty \right) \frac{dt}{t} t^{\theta/2} \mathcal{K}(\mathbf{n}, \mathbf{n}', t). \tag{3.7}$$

When $l \to \infty$, $\mathcal{G}_l^\theta(\mathbf{n}, \mathbf{n}')$ has a singularity as $\mathbf{n} \to \mathbf{n}'$. It comes from the short–time behavior of the heat kernel. Our interest is in this singularity. Thus, the second integral, independent of l, can be discarded and we can set (by an abuse of notation)

$$\mathcal{G}_l^\theta(\mathbf{n}, \mathbf{n}') = \frac{1}{\Gamma(\frac{\theta}{2})} \int_{T(l)}^{T_0} \frac{dt}{t} t^{\theta/2} \mathcal{K}(\mathbf{n}, \mathbf{n}', t). \tag{3.8}$$

At short times, the heat kernel has the asymptotic expansion [18]

$$\mathcal{K}(\mathbf{n}, \mathbf{n}', t) = \frac{\Delta(\mathbf{n}, \mathbf{n}')}{(4\pi t)^{d/2}} \mathrm{e}^{-\frac{\sigma(\mathbf{n}, \mathbf{n}')}{4t}} \left[\mathcal{I} + \sum_{n=1}^{N} \mathcal{A}_n(\mathbf{n}, \mathbf{n}') t^n \right] + \mathcal{O}(t^{N+1}), \tag{3.9}$$

where $\mathcal{I}$ is the identity operator, and the functions Δ, σ and A_n are independent of t and have the short distance behavior

$$\Delta(\mathbf{n},\mathbf{n}') \to 1, \ \sigma(\mathbf{n},\mathbf{n}') \to |\mathbf{n}-\mathbf{n}'|^2 \text{ and } \mathcal{A}_n(\mathbf{n},\mathbf{n}') \to \mathcal{B}_{2n}(\mathbf{n}), \tag{3.10}$$

when $\mathbf{n} \to \mathbf{n}'$. B_{2n} are the Seeley coefficients for the Dirac operator and $\sigma(\mathbf{n},\mathbf{n}')$ is the geodesic distance from $\mathbf{n}$ to $\mathbf{n}'$.

The corresponding asymptotic expansion of $\mathcal{G}_l^\theta$ is thus

$$\mathcal{G}_l^\theta(\mathbf{n},\mathbf{n}') = \frac{\Delta(\mathbf{n},\mathbf{n}')}{(4\pi)^{d/2}\Gamma(\frac{\theta}{2})}\int_{T(l)}^{T_0}\frac{dt}{t}t^{\frac{\theta-d}{2}}\mathrm{e}^{-\frac{\sigma(\mathbf{n},\mathbf{n}')}{4t}}\left[\mathcal{I}+\sum_{n=1}^{N}\mathcal{A}_n(\mathbf{n},\mathbf{n}')t^n\right]+\mathcal{O}(t^{N+1}). \tag{3.11}$$

The leading behavior of this expression as $l \to \infty$, or equivalently $T(l) \to 0$, and $|\mathbf{n}-\mathbf{n}'| \to 0$ can be found by letting T_0 also go to ∞: this only adds terms well behaved as $\mathbf{n} \to \mathbf{n}'$. We find

$$\mathcal{G}_l^\theta(\mathbf{n},\mathbf{n}') = \frac{2^{d-\theta}\Delta(\mathbf{n},\mathbf{n}')}{(4\pi)^{d/2}\Gamma(\frac{\theta}{2})}\left[\frac{\Gamma(\frac{d-\theta}{2})\mathcal{I}}{(\sigma(\mathbf{n},\mathbf{n}'))^{\frac{d-\theta}{2}}}+\sum_{n=1}^{N}\frac{\Gamma(\frac{d-\theta-2n}{2})\mathcal{A}_n(\mathbf{n},\mathbf{n}')}{2^{2n}(\sigma(\mathbf{n},\mathbf{n}'))^{\frac{d-\theta-2n}{2}}}\right]. \tag{3.12}$$

For the fuzzy sphere S_F^2, this gives for the dominant term of $\mathcal{G}_l(\mathbf{n},\mathbf{n}')$, i.e. $\mathcal{G}_l^1(\mathbf{n},\mathbf{n}')$, the expression

$$\mathcal{G}_l(\mathbf{n},\mathbf{n}') \simeq \frac{1}{2\pi}\frac{\mathcal{I}}{|\mathbf{n}-\mathbf{n}'|} \tag{3.13}$$

so that the singular part of

$$\varepsilon(\mathbf{n},\mathbf{n}') := <\mathbf{n}|\varepsilon|\mathbf{n}'> = \frac{1}{2\pi}\mathcal{D}_{\mathbf{n}}\left(\frac{\mathcal{I}}{|\mathbf{n}-\mathbf{n}'|}\right), \tag{3.14}$$

where the differentiation $\mathcal{D}_{\mathbf{n}}$ acts on the variable $\mathbf{n}$.

In the same way, for $d=4$, the singular part is

$$\varepsilon(\mathbf{n},\mathbf{n}') = \frac{1}{4\pi^2}\mathcal{D}_{\mathbf{n}}\left(\frac{\mathcal{I}}{|\mathbf{n}-\mathbf{n}'|^3}+\frac{B_2(\mathbf{n})}{2|\mathbf{n}-\mathbf{n}'|}\right). \tag{3.15}$$

We will show that the second term, although divergent, does not contribute to the continuum limit. In d–dimensions, the corresponding expression is

$$\varepsilon(\mathbf{n},\mathbf{n}') = \frac{1}{(2\pi)^{d/2}}\sum_{n=0}^{d/2-1}\frac{(d-3-2n)!!B_{2n}}{2^n|\mathbf{n}-\mathbf{n}'|^{d-1-2n}}, \tag{3.16}$$

where $B_0=1$, $(-1)!!=1$, and d must be even. If d is odd, the factors and $\Gamma(\frac{d-\theta-2n}{2})$ can have a pole for $\theta=1$. This indicates that the cutoffs $T(l)\to 0$ an $T_0 \to 0$ should be treated with more care. The terms with $n<(d-1)/2$ are not problematic and yield the leading singularities, while the term $n=(d-1)/2$ does not in fact yield a singularity as $\mathbf{n}\to\mathbf{n}'$. Hence for odd d we have

$$\varepsilon(\mathbf{n},\mathbf{n}') = \frac{1}{2\pi^{d/2}}\sum_{n=0}^{\frac{d-1}{2}-1}\frac{(\frac{d-1}{2}-n+1)!B_{2n}}{2^{2n}|\mathbf{n}-\mathbf{n}'|^{d-1-2n}}, \tag{3.17}$$

IV. ELEMENTARY QUANTITIES ON FUZZY SPACES

Here we examine the asymptotics of certain basic expressions on the fuzzy sphere.

Consider first

$$v_2 = \mathrm{Tr}_{\mathcal{H}^l}(\frac{1}{|D|^2}). \tag{4.1}$$

The eigenvalues of D are $\pm\lambda_j = \pm(j+1/2)$ for $j < 2l+1/2$, and $\lambda_{2l+1/2} = +(2l+1)$ for $j = 2l+1/2$, each with multiplicity $(2j+1)$. Thus

$$v_2 = 2\sum_{j=1/2}^{2l+1/2} \frac{2j+1}{(j+1/2)^2} - \frac{2}{2l+1} = 4\psi(2l+1) - 4\psi(1) - \frac{2}{2l+1} \tag{4.2}$$

where $\psi(x) = d\ln\Gamma(x)/dx$. For large l,

$$v_2 = 4\ln(2l+1) + 4\gamma_E - \frac{4}{(2l+1)} + \ldots,$$

with γ_E the Euler constant. Thus

$$\lim_{l\to\infty}\frac{\pi}{\ln l} v_2 = 4\pi = \text{Volume of } (S^2). \tag{4.3}$$

A slight generalization of this quantity in d dimensions is

$$w(\phi) = \frac{(4\pi)^{d/2}\Gamma(\frac{d}{2})}{2d_\gamma \ln(\Lambda(l))}\mathrm{Tr}_{\mathcal{H}^l}\left(\frac{1}{|D|^d}\phi\right) \tag{4.4}$$

where ϕ behaves like a smooth function on the manifold for large l. For large l then, using the Appendix,

$$w(\phi) \sim \frac{(4\pi)^{d/2}\Gamma(\frac{d}{2})}{2d_\gamma \ln(\Lambda(l))}\int_{\mathcal{M}} d^d\mathbf{n}\ \mathrm{tr}[\mathcal{G}_l^\theta(\mathbf{n},\mathbf{n}')]\phi(\mathbf{n}) \tag{4.5}$$

where tr with lower case 't' indicates a trace over the γ–matrices and d_γ is the dimension of the γ–matrices in D. This is

$$w(\phi) \sim \int_V d^d\mathbf{n}\phi(\mathbf{n}). \tag{4.6}$$

This is an important result and indicates how a potential (typically a polynomial in fields) added to our free actions will have the usual continuum limit. It also gives the asymptotic form

$$\frac{1}{g_l} = \frac{(4\pi)^{d/2}\Gamma(\frac{d}{2})}{2d_\gamma \ln(\Lambda(l))} \tag{4.7}$$

V. THE SCALAR ACTION

We rewrite the scalar action (2.13) as

$$S(\phi) = \frac{1}{g_l}\mathrm{Tr}_{\mathcal{H}^l}\left(\frac{1}{|D|^{d-2}}[\varepsilon,\phi]^\dagger[\varepsilon,\phi]\right). \tag{5.1}$$

Its behavior for large l can be deduced from that of $S_l(\phi_0,\phi_1,\ldots,\phi_n)$ in (2.18). We consider the latter for $n=3$, the critical dimension $d=2$, and the non–critical dimension $d\neq 2$ separately. The treatment for a general n is similar but presents extra algebraic complexities.

A. The critical dimension

The behavior of $S(\phi)$ as $l \to \infty$ can be deduced from that of S_l in (2.18) for $p = 0$ and $n = 2$. For large l, we get

$$S_l(\phi_0, \phi_1, \phi_2) = \mathrm{tr} \int d^2\mathbf{n}_1 \ d^2\mathbf{n}_2 \phi_0(\mathbf{n}_1) < \mathbf{n}_1 | [\varepsilon_l, \phi_1] | \mathbf{n}_2 > < \mathbf{n}_2 | [\varepsilon_l, \phi_2] | \mathbf{n}_1 > . \tag{5.2}$$

Using Eq. (2.24) where the kernel $\mathcal{G}_l$ is given in (3.11), one finds in the leading term in large l

$$S_l(\phi_0, \phi_1, \phi_2) = \int d^2\mathbf{n}_1 \ d^2\mathbf{n}_2 \phi_0(\mathbf{n}_1) \mathcal{D}_{\mathbf{n}_1} \mathcal{G}_l(\mathbf{n}_1, \mathbf{n}_2) \left\{ (\phi_1(\mathbf{n}_1) - \phi_1(\mathbf{n}_2)) \right\} \tag{5.3}$$

$$\times \mathcal{D}_{\mathbf{n}_2} \mathcal{G}_l(\mathbf{n}_2, \mathbf{n}_1) \left\{ (\phi_2(\mathbf{n}_2) - \phi_2(\mathbf{n}_1)) \right\} . \tag{5.4}$$

We are interested in the (logarithmically) divergent term in this expression for $l \to \infty$ which comes from the coincidence limit of $\mathcal{G}_l$. This suggests the change of variables

$$\xi = \mathbf{n}_1 - \mathbf{n}_2 \tag{5.5}$$

$$\overline{\mathbf{x}} = \frac{\mathbf{n}_1 + \mathbf{n}_2}{2} \tag{5.6}$$

with the expansions

$$\phi(\mathbf{n}_1) = \phi(\overline{\mathbf{x}} + \frac{\xi}{2}) = \phi(\overline{\mathbf{x}}) + \frac{\xi^i}{2} \partial_i \phi(\overline{\mathbf{x}}) + \mathcal{O}_s(\xi^2) \tag{5.7}$$

$$\phi(\mathbf{n}_2) = \phi(\overline{\mathbf{x}} - \frac{\xi}{2}) = \phi(\overline{\mathbf{x}}) - \frac{\xi^i}{2} \partial_i \phi(\overline{\mathbf{x}}) + \mathcal{O}_s(\xi^2) \tag{5.8}$$

$$\phi(\mathbf{n}_1) - \phi(\mathbf{n}_2) = \xi^i \partial_i \phi(\overline{\mathbf{x}}) + \mathcal{O}_s(\xi^2), \tag{5.9}$$

valid for any field ϕ. Hence

$$\mathcal{D}_{\mathbf{n}_1} \mathcal{G}_l(\mathbf{n}_1, \mathbf{n}_2) \left\{ (\phi_1(\mathbf{n}_1) - \phi_1(\mathbf{n}_2)) \right\} = \left\{ (\mathcal{D}_\xi \mathcal{G}_l(|\xi|)) \xi^i \right\} \partial_i \phi(\overline{\mathbf{x}}) + \mathcal{O}(|\xi|^1). \tag{5.10}$$

Putting all this together and using polar coordinates, for which $d^2\xi = |\xi| d|\xi| d\Omega$, we find for l large

$$S_l(\phi_0, \phi_1, \phi_2) = N_l^{ij} \int d^2\overline{\mathbf{x}} \left[\phi_0(\overline{\mathbf{x}}) \partial_i \phi_1(\overline{\mathbf{x}}) \partial_j \phi_2(\overline{\mathbf{x}}) + \mathcal{O}_{ij}(|\xi|) \right], \tag{5.11}$$

with

$$N_l^{ij} = \mathrm{tr} \int d|\xi| |\xi| d\Omega \left\{ \mathcal{D}_{\xi_k} \mathcal{G}_l(|\xi|) \mathcal{D}_{\xi_k} \mathcal{G}_l(|\xi|) \right\} \xi^i \xi^j. \tag{5.12}$$

By rotational invariance, $N_l^{ij} = N_l \delta^{ij}$. Hence we get

$$N_l = 2\pi \int_0^\infty d|\xi| |\xi|^3 \partial_k \mathcal{G}_l(|\xi|) \partial^k \mathcal{G}_l(|\xi|). \tag{5.13}$$

The expansion (3.11) shows that only the leading term contributes to the logarithmic divergence. To isolate this divergence, we use the leading term and find

$$N_l = \frac{2}{\pi^2} \int_{T(l)}^{T_0} dt_1 \int_{T(l)}^{T_0} dt_2 \frac{\sqrt{t_1 t_2}}{(t_1 + t_2)^3}. \tag{5.14}$$

The indefinite form of this integral is given by polylogarithms

$$Li_a(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^a} \tag{5.15}$$

in the form

$$\int dxdy \frac{\sqrt{xy}}{(x+y)^3} = \frac{1}{2}\frac{\sqrt{xy}}{(x+y)} + \frac{i}{4}\left\{ Li_2(i\sqrt{\frac{x}{y}}) - Li_2(-i\sqrt{\frac{x}{y}}) \right\}. \tag{5.16}$$

The asymptotic behavior of the polylogarithm for small z is

$$Li_2(-i/z) = -1/2(\ln z)^2 - i\frac{\pi}{2}\ln z - \frac{\pi^2}{24} + \dots. \tag{5.17}$$

Thus

$$N_l = \frac{1}{2\pi}\ln l + \mathcal{O}(1). \tag{5.18}$$

Hence in the critical dimension 2,

$$\lim_{l\to\infty} \frac{\pi}{\ln(l)} S_l(\phi_0,\phi_1,\phi_2) = \frac{1}{2}\int_{S^2} d^2\mathbf{n}\ \phi_0(\mathbf{n})\partial_i\phi_1(\mathbf{n})\partial^i\phi_2(\mathbf{n}). \tag{5.19}$$

It is the local scalar action once obvious choices of ϕ_i are made.

B. The non–critical dimensions

When $d > 2$, there is no need to approximate ε by ε_l as in (2.18): the Appendix shows that it has the same leading large l behavior as (2.18). Instead, we can redefine S_l to be

$$S_l(\phi_0,\phi_1,\phi_2) = \mathrm{Tr}(\frac{1}{|D_l|^{d-2}}\phi_0[\varepsilon,\phi_1][\varepsilon,\phi_2]), \tag{5.20}$$

For large l then,

$$S_l(\phi_0,\phi_1,\phi_2) = \int d^d\mathbf{n}_1\ d^d\mathbf{n}_2\ d^d\mathbf{n}_3 \mathcal{G}_l^{d-2}(\mathbf{n}_1,\mathbf{n}_2)\phi_0(\mathbf{n}_2)\mathcal{D}_{\mathbf{n}_2}\mathcal{G}(\mathbf{n}_2,\mathbf{n}_3)\left\{(\phi_1(\mathbf{n}_3)\right. \tag{5.21}$$

$$\left. -\phi_1(\mathbf{n}_2))\right\}\mathcal{D}_{\mathbf{n}_3}\mathcal{G}(\mathbf{n}_3,\mathbf{n}_1)\left\{(\phi_2(\mathbf{n}_1) - \phi_2(\mathbf{n}_3))\right\},\ d > 2. \tag{5.22}$$

The kernels appearing here are divergent when $|\mathbf{n}_i - \mathbf{n}_j| \to 0$ and give rise to the overall logarithmic divergence. The following change of variable helps to isolate them:

$$\xi_1 = \mathbf{n}_3 - \mathbf{n}_2 \tag{5.23}$$

$$\xi_2 = \mathbf{n}_3 - \mathbf{n}_1 \tag{5.24}$$

$$\overline{\mathbf{x}} = \frac{\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3}{3}. \tag{5.25}$$

We then find as $l \to \infty$,

$$S_l(\phi_0,\phi_1,\phi_2) = N_l^{ij}\int d^d\overline{\mathbf{x}}\phi_0(\overline{\mathbf{x}})\partial_i\phi_1(\overline{\mathbf{x}})\partial_j\phi_2(\overline{\mathbf{x}}) \tag{5.26}$$

where, by spherical symmetry,

$$N_l^{ij} = N_l\delta^{ij}. \tag{5.27}$$

We find

$$N_l = d_\gamma \int d\xi_1 d\xi_2 (\xi_1 \cdot \xi_2) \mathcal{G}_a^{(d-2)}(|\xi_1 + \xi_2|) \partial_i \mathcal{G}^1(|\xi_1|) \partial^i \mathcal{G}(|\xi_2|) \tag{5.28}$$

$$= \frac{4d_\gamma}{(4\pi)^{\frac{d+2}{2}} \Gamma(\frac{d-2}{2})} \int_{1/T_0}^{1/T(l)} du \int_0^{+\infty} du_1 \int_0^{+\infty} du_2 \Big((1 + u(u_1^2 + u_2^2))^{-\frac{d+2}{2}} \tag{5.29}$$

$$+ (d+2) u^2 u_1^2 u_2^2 (1 + u(u_1^2 + u_2^2))^{\frac{d+4}{2}} \Big), \tag{5.30}$$

where we made use of the leading terms of Eqs. (3.11) and (3.16). The integrations over u_1 and u_2 can be done by performing a double integration by parts on the second term and then going to polar coordinates. The last integral, over u, isolates the logarithmic divergence,

$$N_l \approx \frac{2d_\gamma \ln(l)}{(4\pi)^{d/2} \Gamma(\frac{d}{2})} \frac{d-1}{d}. \tag{5.31}$$

It reduces to (5.18) in the limit of the critical dimension $d = 2$.

Thus, for the general d,

$$\lim_{l \to \infty} \frac{(4\pi)^{d/2} \Gamma(\frac{d}{2})}{2d_\gamma \ln(\Lambda)} \varphi_a(\phi_1, \phi_2, \phi_3) = \frac{d-1}{d} \int_V d^d \mathbf{n} \phi_1(\mathbf{n}) \partial_i \phi_2(\mathbf{n}) \partial^i \phi(\mathbf{n}). \tag{5.32}$$

VI. THE SPINOR ACTION

As $d \geq 2$, we do not encounter the critical dimension $d = 1$ of the spinor action $S(\psi)$. It is enough for us to study $S_l(\psi_1, \psi_2)$ for large l, as the limiting form of $S(\psi)$ can be deduced therefrom.

In the coherent state representation, S_l reads

$$S_l(\psi_1, \psi_2) = \mathrm{tr} \left(\int d^d \mathbf{n}_1 d^d \mathbf{n}_2 \mathcal{G}_l^{d-1}(\mathbf{n}_1, \mathbf{n}_2) \psi_1^\dagger(\mathbf{n}_2) \varepsilon(\mathbf{n}_2, \mathbf{n}_1) \psi_2(\mathbf{n}_1) \right), \tag{6.1}$$

where we have replaced ε_l by ε, and the spinors are assumed to be local in the continuum limit:

$$< \mathbf{n} | (\psi_i)_r | \mathbf{n}' > = (\psi_i)_r(\mathbf{n}) \delta(\mathbf{n} - \mathbf{n}'). \tag{6.2}$$

It is easy to check the presence of a logarithmic divergence as $|\mathbf{n}_1 - \mathbf{n}_2| \to 0$. We can make it explicit by changing to the variables $(\xi, \overline{\mathbf{x}})$ described in (5.6):

$$S_l(\psi_1, \psi_2) = N_l^{ij} \left(\frac{\gamma_j}{2} \int d\overline{\mathbf{x}} (\psi_1(\overline{\mathbf{x}}) \partial_i \psi_2(\overline{\mathbf{x}}) - \partial_i \psi_1(\overline{\mathbf{x}}) \psi_2(\overline{\mathbf{x}})) \right), \tag{6.3}$$

$$N_l^{ij} = -\delta^{ij} \frac{d_\gamma}{d} \int d^d \xi \; \mathcal{G}_l^{d-1}(|\xi|) \xi^i \varepsilon_i(\xi). \tag{6.4}$$

Using the leading terms of Eqs. (3.11) and (3.16), and going to spherical coordinates, we can evaluate N_l^{ij} to get

$$N_l^{ij} = \delta^{ij} \frac{d-1}{d} \frac{2d_\gamma \ln(l)}{(4\pi)^{d/2} \Gamma(\frac{d}{2})} \tag{6.5}$$

so that

$$\lim_{l \to \infty} \frac{(4\pi)^{d/2} \Gamma(\frac{d}{2})}{2d_\gamma \ln(l)} S_l(\psi_1, \psi_2) = \frac{d-1}{d} \int d^d \mathbf{n} \psi_1^\dagger(\mathbf{n}) \mathcal{D} \psi_2(\mathbf{n}), \tag{6.6}$$

which, up to the normalization, is the continuum Dirac action.

VII. CONCLUSION

The fuzzy actions of [13,14] where motivated by topological considerations. From their appearance, it is not obvious that they have the desired continuum limits. In this paper, we have established that these limits can be achieved for the scalar and spinor actions if they are suitably scaled when the cut–off l approaches ∞. The scaling factors are proportional to $(\ln(l))^{-1}$.

It remains to extend these considerations to actions of gauge theories and to study the partition and correlation functions. The latter, at least for the "free field" models considered here should be accessible to analytic methods, these and the gauge field actions should have the usual continuum limits. The study of the rate of approach to these limits would be of particular interest.

ACKNOWLEDGMENTS

This work was supported by the joint NSF-CONACyT grant number $E120.0462/2000$. A.P. Balachandran and D. O'Connor were also supported respectively by the DOE grant $DE-FG02-85ER40231$, and the CONACyT grant $30422-E$. X. Martin was supported by CONACyT and SNI-México.

APPENDIX

In this Appendix will be explained how the functionals on the non-commutative spaces whose limits are to be studied can be turned into integrals on the limiting commutative manifold. For that, it will be convenient to suppose that each of the non–commutative algebras M_l admit a coherent states representation.

Coherent states form a complete set of vectors $|\mathbf{n}, l>$ on the algebra indexed by points $\mathbf{n}$ on the limiting manifold $\mathcal{M}$. They have the asociated resolution of the identity

$$\int_{\mathcal{M}} d^d\mathbf{n} |\mathbf{n}, l><\mathbf{n}, l| = 1 \tag{7.1}$$

and are orthogonal in the limit $l \to \infty$:

$$<\mathbf{n}, l|\mathbf{n}', l>= \delta^d(\mathbf{n} - \mathbf{n}') + o(1). \tag{7.2}$$

This coherent state representation naturally generates a linear imbedding of the non-commutative algebras M_l into the algebra of functions on $\mathcal{M}$:

$$\varphi_l \in M_l \mapsto \varphi(\mathbf{n}) =<\mathbf{n}, l|\varphi_l|\mathbf{n}, l> . \tag{7.3}$$

In general, and that is certainly true in the case of a compact simple Lie group, this mapping is injective. This is because the coherent states form an overcomplete set, so that the diagonal terms $\varphi(\mathbf{n})$ are sufficient to reconstruct completely the initial matrix φ_l. In the following, we will assume that it *is* possible to reconstruct any field from its diagonal elements in the coherent state basis.

This mapping is not a mapping of algebras since it can not map a non-commutative multiplication to a commutative one. The product obtained on the algebra of functions by mapping the product on the non-commutative algebras will be called $*$–product. It is thus defined as

$$<\mathbf{n}, l|\varphi_l\psi_l|\mathbf{n}, l>= \varphi(\mathbf{n}) * \psi(\mathbf{n}). \tag{7.4}$$

With the assumption above, the $*$–product goes over to the normal product on the manifold in the commutative limit $l \to \infty$, with corrections which go like $1/l$. In a way, this property means that fields become local in the continuum limit.

Something similar can also be derived with operators other than multiplication. Indeed, if $\mathcal{O}_l$ is an operator on the Hilbert space $\mathcal{H}_l$, it is mapped to a kernel

$$\mathcal{O}(\mathbf{n}, \mathbf{n}') = <\mathbf{n}, l|\mathcal{O}|\mathbf{n}', l> \tag{7.5}$$

on the manifold $\mathcal{M}$. In this case, there is no particular reason to think that the diagonal elements of the operator should be sufficient to describe it completely. Then composing this operator with a left multiplication operator will map to another kind of product which will be denoted as #

$$<\mathbf{n}|\mathcal{O}\phi_l|\mathbf{n}'> = \mathcal{O}(\mathbf{n}, \mathbf{n}')\#\phi(\mathbf{n}'). \tag{7.6}$$

It is a reasonable assumption, which will be made in the following, that this product also goes over to the standard product in the large l limit, with corrections of order $1/l$.

At this point, taking for instance the functional (2.17) with $n = 2$, one can write

$$S_l(\psi_1, \psi_2) = \frac{1}{g_l}\mathrm{Tr}_{\mathcal{H}^l}\left(\frac{1}{|D|^p}\psi_1\dagger\epsilon\psi_2\right) \tag{7.7}$$

$$\sim \frac{1}{g_l}\int_{\mathcal{M}} d^d\mathbf{n}_1 d^d\mathbf{n}_2 <\mathbf{n}_1, l|\frac{1}{|D|^p}|\mathbf{n}_2, l> \#\psi_1^\dagger(\mathbf{n}_2)\# <\mathbf{n}_2, l|\epsilon|\mathbf{n}_1, l> \#\psi_2(\mathbf{n}_1), \tag{7.8}$$

where the sequence g_l is expected to diverge logarithmically to get a finite result for the continuum action, and $\sim$ means that the two expressions are equivalent when $l \to \infty$.

Because of the ansatz on the limit of the #–product made above, it should be clear that the expected logarithmic divergence of (7.8) will not be affected if #–products are replaced by standard products. This yields immediately

$$S_l(\psi_1, \psi_2) \sim \frac{1}{g_l}\int_{\mathcal{M}} d^d\mathbf{n}_1 d^d\mathbf{n}_2 <\mathbf{n}_1, l|\frac{1}{|D|^p}|\mathbf{n}_2, l> \psi_1^\dagger(\mathbf{n}_2) <\mathbf{n}_2, l|\epsilon|\mathbf{n}_1, l> \psi_2(\mathbf{n}_1). \tag{7.9}$$

At this point, what remains to be done is to evaluate the limiting behaviour of the kernels for $1/|D|^p$ and ϵ which appear on the right hand side.

It is reasonable to think that, having the same eigenvalues and eigenfunctions with the same structure, the discrete operators $1/|D|^p$ and ϵ should converge to their equivalent counterparts $1/|\mathcal{D}_c|^p$ and ε_c whose eigenvalues larger than the cut–off eigenvalue $\Lambda(l)$ have been set to zero. Thus,

$$S_l(\psi_1, \psi_2) \sim \frac{1}{g_l}\int_{\mathcal{M}} d^d\mathbf{n}_1 d^d\mathbf{n}_2 <\mathbf{n}_1|\frac{1}{|\mathcal{D}_c|^p}|\mathbf{n}_2> \psi_1^\dagger(\mathbf{n}_2)\varepsilon_c(\mathbf{n}_2, \mathbf{n}_1)\psi_2(\mathbf{n}_1). \tag{7.10}$$

These truncated operators in the continuum are still difficult to describe since their cut–off is spectral. However, in the limit $l \to \infty$, these truncated operators converge weakly (i.e their action on any given spinor converges to the action of the weak limit) to the usual Dirac kernels $1/|\mathcal{D}|^p$ and ε. This is simply because these operators are bounded and therefore their high frequency behaviour is irrelevant. Indeed, calling generically $\mathcal{O}$ and $\mathcal{O}_c$ the operator considered and its truncation respectively,

$$(\mathcal{O} - \mathcal{O}_c)(\sum_{i=0}^{\infty} c^j_{ms}E^j_{ms}) = \mathcal{O}(\sum_{i=\Lambda(l)}^{\infty} c^j_{ms}E^j_{ms}) \leq |||\mathcal{O}|||\ ||\sum_{i=\Lambda(l)}^{\infty} c^j_{ms}E^j_{ms}|| \to 0, \tag{7.11}$$

where $|||\cdot|||$ denotes the norm of the operator $\mathcal{O}$. Note that this shows that operators of the form $[\varepsilon_c, \phi]$ or $\varepsilon_c\psi$ also converge weakly to the corresponding untruncated operators.

This weak convergence is however insufficient for our purpose because we know that the expressions to be studied are logarithmically divergent when $l \to \infty$ and therefore that the high frequency behaviour of the operators *is* important. Still, as long as one truncated operator is kept, the large l behaviour of the others will not matter. Keeping the truncated kernel $1/|\mathcal{D}_c|^p$, we therefore have for the example considered,

$$S_l(\psi_1,\psi_2) = \frac{1}{g_l}\int_{\mathcal{M}} d^d\mathbf{n}_1 d^d\mathbf{n}_2 <\mathbf{n}_1|\frac{1}{|\mathcal{D}_c|^p}|\mathbf{n}_2> \psi_1^\dagger(\mathbf{n}_2)\varepsilon(\mathbf{n}_2,\mathbf{n}_1)\psi_2(\mathbf{n}_1). \tag{7.12}$$

Now for the remaining operator $1/|\mathcal{D}_c|^p$, the high frequency behaviour *does* matter and it will serve to regularise the expressions we are interested in. A good approximation for $1/|\mathcal{D}_c|^p$ is the operator $1/|\mathcal{D}_l|^p$, introduced in (2.18) and defined in (3.4), whose high frequency eigenvalues are made to decrease exponentially. To compare it to the truncated operators, their eigenvalues should be compared. By construction, all these operators are functions of the Dirac operator and therefore have the same eigenfunctions $\mathcal{E}^j_{ms}(\mathbf{n})$, and the eigenvalues Δ_j of their difference are thus just the difference between their respective eigenvalues. We find for the "relative" difference when $E_j \leq \Lambda(l)$:

$$E_j^p\Delta_j = \frac{E_j^p}{\Gamma(\frac{p}{2})}\int_0^{T(l)}\frac{dt}{t}t^{p/2}\mathrm{e}^{-E_j^2 t} \tag{7.13}$$

$$\leq \frac{1}{\Gamma(\frac{p}{2})}\int_0^{T(l)E_j^2}\frac{du}{u}u^{p/2} \leq \frac{1}{\Gamma(\frac{p+2}{2})}\left(\frac{E_j}{\Lambda(l)}\right)^p. \tag{7.14}$$

In the other case $E_j > \Lambda(l)$, the eigenvalues of $1/|\mathcal{D}_c|$ are zero, and the eigenvalues F_j of $1/|\mathcal{D}_l|$ are given by

$$F_j = \frac{1}{\Gamma(\frac{p}{2})}\int_{T(l)}^{+\infty}\frac{dt}{t}t^{p/2}\mathrm{e}^{-E_j^2 t} \tag{7.15}$$

$$\leq \frac{1}{E_j^p\Gamma(\frac{p}{2})}\mathrm{e}^{-T(l)E_j^2/2}\int_0^{+\infty}\frac{du}{u}u^{p/2}\mathrm{e}^{u/2} \leq \frac{2^{p/2}}{E_j^p}\mathrm{e}^{-(E_j/\Lambda(l))^2/2}. \tag{7.16}$$

So, the regularised kernel $1/|\mathcal{D}_l|^p$ is equivalent to the truncated one whenever $E_j \ll \Lambda(l)$, and is exponentially suppressed when $E_j \gg \Lambda(l)$. However, the difference between the two kernels becomes of order one near the cut–off eigenvalue $\Lambda(l)$.

Introducing a complete set of Dirac eigenfunctions in the functional (7.12) and the one we want to replace it by, namely (2.19), we get

$$S_l(\psi_1,\psi_2) \sim \frac{1}{g_l}\sum_{j=0}^{\Lambda(l)}\frac{1}{E_j^p}\int_{\mathcal{M}} d^d\mathbf{n}_1 d^d\mathbf{n}_2 {\mathcal{E}^j_{ms}}^\dagger(\mathbf{n}_2)\psi_1^\dagger(\mathbf{n}_2)\varepsilon(\mathbf{n}_2,\mathbf{n}_1)\psi_2(\mathbf{n}_1)\mathcal{E}^j_{ms}(\mathbf{n}_1), \tag{7.17}$$

$$\mathrm{Tr}_{\mathcal{H}^\infty}\left(\frac{1}{|\mathcal{D}_l|^p}\psi_1^\dagger\varepsilon\psi_2\right) = \frac{1}{g_l}\sum_{j=0}^{\infty}\frac{1}{F_j}\int_{\mathcal{M}} d^d\mathbf{n}_1 d^d\mathbf{n}_2 {\mathcal{E}^j_{ms}}^\dagger(\mathbf{n}_2)\psi_1^\dagger(\mathbf{n}_2)\varepsilon(\mathbf{n}_2,\mathbf{n}_1)\psi_2(\mathbf{n}_1)\mathcal{E}^j_{ms}(\mathbf{n}_1). \tag{7.18}$$

For modes $j \leq \Lambda_-(l) \ll \Lambda(l)$, the inequalities (7.14) suggest that the difference between these two expressions should be subdominant. For modes $j \geq \Lambda(l)$, the inequalities (7.16) suggest that the contribution in Eq. (7.18) should also be subdominant since all modes are exponentially suppressed. Finally for intermediate modes $\Lambda_-(l) \leq j \leq \Lambda(l)$, the relative contribution to the spinor type action in Eq. (7.17) is of order $\ln[\Lambda(l)/\Lambda_-(l)]/\ln(\Lambda(l))$ since we know that it diverges logarithmically in the large l limit. For an appropriate choice of $\Lambda_-(l)$, this relative contribution can be made subdominant. Thus, it is to be expected that the two functionals (7.17) and (7.18) have the same continuum limit $l \to \infty$.

Hence, we have found that, with a sequence g_l which diverges logarithmically,

$$\lim_{l\to\infty}\left(S_l(\psi_1,\psi_2)\right) = \lim_{l\to\infty}\left(\frac{1}{g_l}\int_{\mathcal{M}} d^d\mathbf{n}_1 d^d\mathbf{n}_2 \mathcal{G}_l^p(\mathbf{n}_1,\mathbf{n}_2)\psi_1^\dagger(\mathbf{n}_2)\varepsilon(\mathbf{n}_2,\mathbf{n}_1)\psi_2(\mathbf{n}_1)\right), \tag{7.19}$$

which is exactly the form (2.19) we were looking for. For the other type of functionals, the scalar type actions (2.16), the same reasoning will yield that they are equal to the expression (2.18) we studied in the article.

[1] A. Connes. *Noncommutative Geometry.* Academic Press, London, 1994.

[2] J. Madore. *An Introduction to Noncommutative Differential Geometry and its Applications.* Cambridge University Press, Cambridge, 1995; `gr-qc/9906059`.

[3] G. Landi. *An Introduction to Noncommutative Spaces and Their Geometries.* Springer-Verlag, Berlin, 1997. `hep-th/9701078`.

[4] J. C. Varilly. *An Introduction to Noncommutative Geometry*, `physics/9709045`.

[5] R. Coquereaux. *J. Geo. Phys.* **6**, (1989) 425–490.

[6] J. A. Mignaco, C. Sigaud, A. R. da Silva, and F. J. Vanhecke. *Rev. Math. Phys.* **9**, (1997) 689–718. `hep-th/9611058`.

[7] U. Carow-Watamura and S. Watamura. *Commun. Math. Phys.* **183**, (1997) 365–382. `hep-th/9605003`.

[8] U. Carow-Watamura and S. Watamura. `hep-th/9801195`.

[9] H. Grosse and P. Presnajder. *Lett. Math. Phys.* **33**, (1995) 171–182. and references therein.

[10] H. Grosse, C. Klimcik, and P. Presnajder. *Commun. Math. Phys.* **178**, (1996) 507–526.

[11] H. Grosse, C. Klimcik, and P. Presnajder. *Commun. Math. Phys.* **180**, (1996) 429–438. `hep-th/9602115`.

[12] H. Grosse, C. Klimcik, and P. Presnajder. In *Les Houches Summer School on Theoretical Physics*, 1995. `hep-th/9603071`. See citations in [9–12] for further references.

[13] S. Baez, A. P. Balachandran, B. Idri, and S. Vaidya. *Comm. Math. Phys.* **208**, (2000) 787–798. `hep-th/9811169`.

[14] A. P. Balachandran and S. Vaidya "Instantons and Chiral Anomaly in Fuzzy Physics" `hep-th/9910129`

[15] A. P. Balachandran, T. R. Govindarajan and B. Ydri, "The Fermion Doubling Problem and Noncommutative Geometry" `hep-th/9911087`

[16] J. Frohlich, O. Grandjean, and A. Recknagel. *Commun. Math. Phys.* **203**, (1999) 119–184. `math-phy/9807006` and references therein for fuzzy group manifolds.

[17] J. C. Varilly and J. M. Gracia-Bondia. *J. Geom. Phys.* **12**, (1993) 223–301.

[18] Peter B. Gilkey, "Invariance Theory, The Heat Equation, And the Atiyah-Singer Index Theorem" Mathematics Lecture Series, Vol 11, Publish or Perish, 1984.